## Controls - 1

Cubli, shown in Fig.1a, is a robot developed by ETH Zurich. The robot uses three flywheels for locomotion as well as maintaining its balance when standing on its edges and vertices. The following problem is inspired by the balancing control of Cubli.

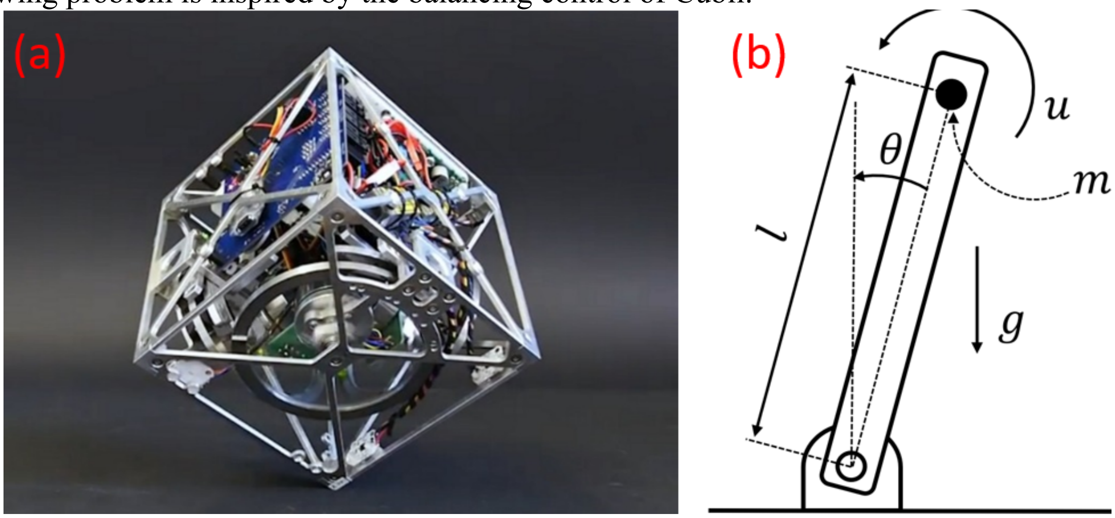



Figure 1: (a) Cubli; (b) Inverted pendulum

Let's consider that Cubli can be modeled as an inverted pendulum (Fig. 1b), whose pivot joint is located on a horizontal plane. For the sake of simplicity assume the mass $m$ of the pendulum is concentrated at the point with a distance $l$ from the pivot joint. The rotation angle $\theta$ reaches $\theta = 0$ when the pendulum is vertical. A torque, $u$ is applied to the pendulum. The gravitational acceleration is $g$.

1) Derive the equation of motion (EOM) of the system. Linearize the EOM by assuming that that the angle $\theta$ is small and present the linearized system in the state-space form with the state vector defined as $x$. **(15 points)**

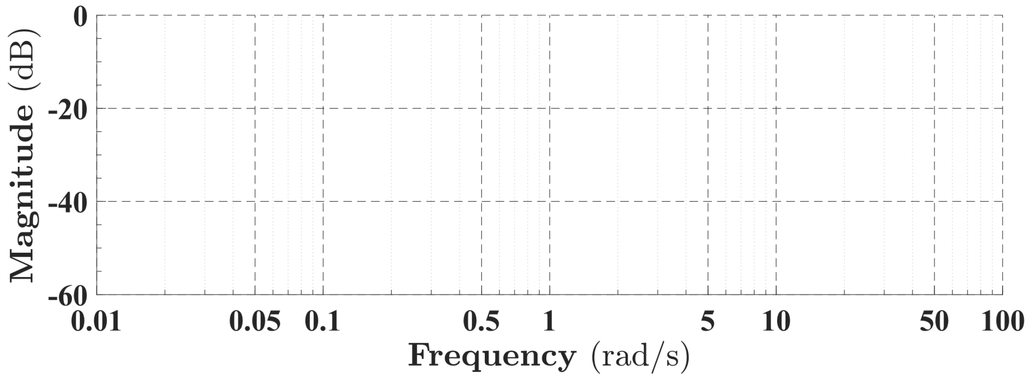
2) Considering that the governing linearized equation of the system is $\ddot{\theta} = \frac{mgl\theta + u}{ml^2}$, where $u$ is the control input and the output $\theta = y$, determine the transfer function $H(s) = Y/U$, where $U = L(u)$ and Y$= L(y)$ (note that $L(\cdot)$ is the Laplace transformation). Find the zeros and poles of the open loop system and determine the stability of the system. What is the physical meaning of the poles of the open loop system? **(20 points)**

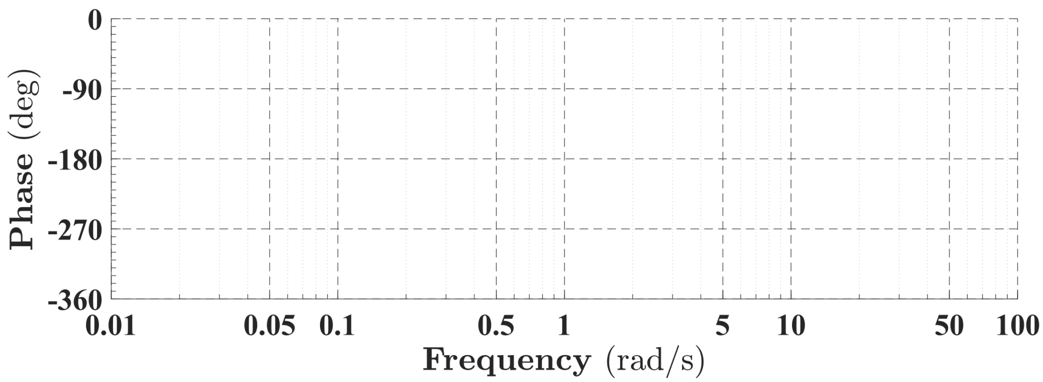
3) Let's consider that a general controller transfer function $G(s)$ will be designed for use in a unity negative feedback loop such that the output $y$ will track a reference input $y_r$. The output of the controller is $U = G(s)(Y_r - Y)$, where $Y_r = L(y_r)$. Draw the closed-loop control system block diagram and calculate the closed-loop transfer function $T(s) = Y/Y_r$. **(20 points)**

**To answer Questions 4 and 5, use the following model parameters:**

$$m = 0.625\ (kg); \quad l = 0.4\ (m); \quad g = 10\ (m/s^2)$$

4) Calculate the frequency response of the open-loop plant $H(s)$ in terms of the magnitude $A$ and phase $\phi$, and sketch the Bode plot in the axes below using accurate asymptotic approximations. From your sketch, determine the 3dB bandwidth (i.e., the cutoff frequency) of the open-loop plant. **(20 points)**

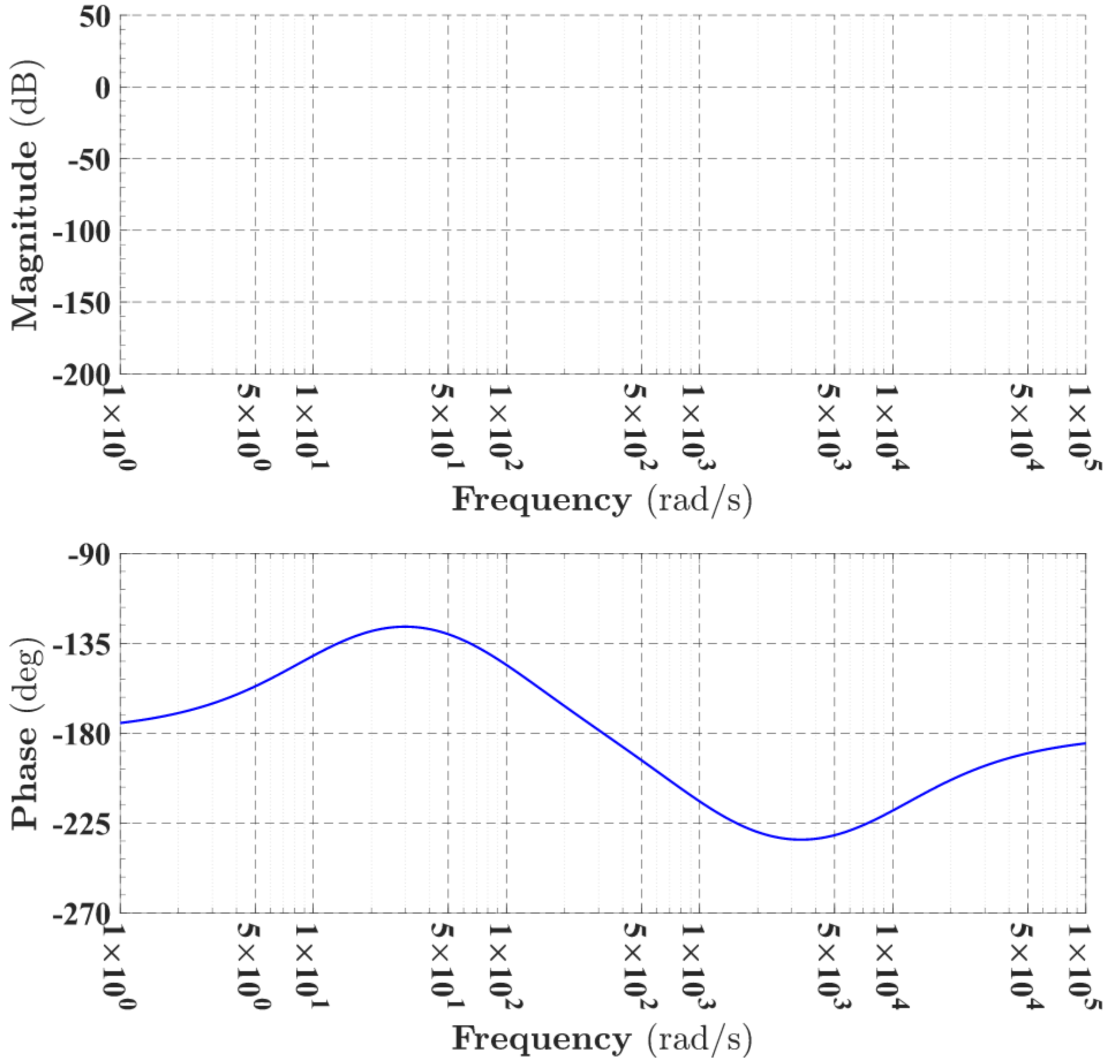



5) Let's consider that *G(s)* is designed as a lead-lag controller so that

$$G(s) = K\frac{(s+10^4)(s+10)}{(s+10^3)(s+10^2)}$$

where $K = 250$ is the gain. Draw the magnitude Bode plot of the open-loop system $G(s)H(s)$ in the axes below. Determine the gain and phase margins from the Bode plots (the phase Bode plot is already provided) and clearly mark on the plot where the grain crossover and phase crossover points are. Discuss the stability of the closed-loop system based on the gain and phase margins. What happens to the stability of the closed-loop system when $K$ increases? **(25 points)**


50
0
-50
-100
-150
-200
Magnitude (dB)
1×10^0
5×10^0
1×10^1
5×10^1
1×10^2
5×10^2
1×10^3
5×10^3
1×10^4
5×10^4
1×10^5
Frequency (rad/s)
-90
-135
-180
-225
-270
Phase (deg)
1×10^0
5×10^0
1×10^1
5×10^1
1×10^2
5×10^2
1×10^3
5×10^3
1×10^4
5×10^4
1×10^5
Frequency (rad/s)